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TWO ELEMENTARY GEOMETRICAL APPLICATIONS OF DETERMINANTS.

BY H. S. WHITE.

IN attacking a geometrical problem by analytical methods we often have two courses open to us.

1. We may endeavor to simplify the problem as far as possible by a special choice of the coordinate system. This is deservedly a favorite method, but it usually involves a loss of symmetry which may overbalance the simplification. It not infrequently has the further disadvantage of making necessary a separate consideration of exceptional cases, a distinction irrelevant to the main problem having been introduced by the special coordinate system.

2. We may attempt to give to the analytical work the utmost possible symmetry. This method often calls for skill in the manipulation of algebraical expressions, and, in particular, readiness in the use of determinants.

A student who goes beyond the elements of geometrical study will soon find himself obliged to make use of the second of these methods, *e. g.* in the study of line geometry in three dimensions. It is well for him therefore to have practice in this kind of work, even in the solution of elementary problems where the first method is ordinarily used. With this idea in mind, the following symmetrical treatment of Desargues' and Pascal's theorems has been prepared. Incidentally attention is called (§ 3) to a rarely noticed natural connection between these theorems, the two sets of alternate sides of an inscribed hexagon constituting two triangles in perspective position, whose vertices suggest at once Brianchon's hexagon.

1. Notation and a Reducing Formula. Let a point be denoted by homogeneous coordinates, as x_1, x_2, x_3; a line by u_1, u_2, u_3; the triangle of reference being the same for both systems, and the two systems so chosen that when the line contains the point the equation is satisfied:

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0.$$

If two points (p_1, p_2, p_3) and (q_1, q_2, q_3) lie on a line (u_1, u_2, u_3), the two equations yield the proportions: $u_1 : u_2 : u_3 = (p_2 q_3) : (p_3 q_1) : (p_1 q_2)$. Denote these two-rowed determinants by $(p\,q)_1$, $(p\,q)_2$, $(p\,q)_3$ respectively.

Similarly if two lines (u_1, u_2, u_3) and (v_1, v_2, v_3) determine a point (x_1, x_2, x_3), we find the proportions:

$$x_1 : x_2 : x_3 = (u_2 v_3) : (u_3 v_1) : (u_1 v_2) = (u\,v)_1 : (u\,v)_2 : (u\,v)_3.$$

Accordingly if two points (a) and (b) determine a line (u), and two others, (c) and (d), the line (v), while (u) and (v) meet in a point (x), there will be no obscurity in the following abbreviation:

$$\begin{aligned} x_1 : x_2 : x_3 &= (u\,v)_1 : (u\,v)_2 : (u\,v)_3 \\ &= ((a\,b)_2\,(c\,d)_3) : ((a\,b)_3\,(c\,d)_1) : ((a\,b)_1\,(c\,d)_2) \\ &= (a\,b,\,c\,d)_1 : (a\,b,\,c\,d)_2 : (a\,b,\,cd)_3. \end{aligned}$$

For a particular example, suppose that the points (a) and (c) coincide, so that the point (x) is the point (a). This is shown from the above proportions by substituting for c_1, c_2, c_3 respectively a_1, a_2, a_3, so that we have

$$x_1 : x_2 : x_3 = [(a\,b)_2\,(a\,d)_3] : [(a\,b)_3\,(a\,d)_1] : [(a\,b)_1\,(a\,d)_2],$$

and then noticing that these are, save for sign, first minors of the determinant

$$D' = \begin{vmatrix} (ba)_1 & (ba)_2 & (ba)_3 \\ (ad)_1 & (ad)_2 & (ad)_3 \\ (db)_1 & (db)_2 & (db)_3 \end{vmatrix},$$

which is the conjugate or "reciprocal" of the determinant:

$$D = \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}.$$

Accordingly these minors are proportional to their complementary constituents in D, and are respectively equal to the following:

$$D \cdot a_1, \quad D \cdot a_2, \quad D \cdot a_3.$$

It will be necessary to reduce to simpler form several determinants whose constituents are themselves two-rowed determinants, like the following:

$$R = \begin{vmatrix} (ab)_1 & (cd)_1 & (xy)_1 \\ (ab)_2 & (cd)_2 & (xy)_2 \\ (ab)_3 & (cd)_3 & (xy)_3 \end{vmatrix}.$$

This may be expanded by distributing — so to term it — the constituents of any one column. To distribute the third column, assume any arbitrary quantities z_1, z_2, z_3, and multiply R by the determinant $(x\,y\,z)$, column-wise:

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \cdot R = \begin{vmatrix} (abx) & (cdx) & (xyx) \\ (aby) & (cdy) & (xyy) \\ (abz) & (cdz) & (xyz) \end{vmatrix}$$

$$= \begin{vmatrix} (abx) & (cdx) \\ (aby) & (cdy) \end{vmatrix} \cdot (xyz).$$

For (xyx) and (xyy) are identically zero. Omitting the factor (xyz) on each side of the above equation, we have the reduction desired:

$$R \equiv \Big((a\,b)_1\ \ (c\,d)_2\ (x\,y)_3\Big) = (a\,b\,x)\ (c\,d\,y) - (a\,b\,y)\ (c\,d\,x).$$

Of course also the first or the second column could have been distributed, giving the alternative reductions:

$$\begin{aligned} R &= (a\,c\,d)\ (b\,x\,y) - (b\,c\,d)\ (a\,x\,y) \\ &= (a\,b\,d)\ (c\,x\,y) - (a\,b\,c)\ (d\,x\,y). \end{aligned}$$

2. Two perspective triangles,—Desargues' Theorem.

If two triangles $A\,B\,C$ and $D\,E\,F$ are so situated that lines joining corresponding vertices $A\,D$, $B\,E$, $C\,F$ meet in a point, then pairs of corresponding sides meet in three points of a right line.

The proof will be made by examining the condition $P=0$ for three lines meeting in a point, and the condition $L=0$ for three points lying on a line, and showing that P is a factor of L. Denoting the coordinates of A by a_1, a_2, a_3, etc., we have these two conditions in a form to which the reduction in §1 can be applied, as follows:

$$P = \begin{vmatrix} (ad)_1 & (be)_1 & (cf)_1 \\ (ad)_2 & (be)_2 & (cf)_2 \\ (ad)_3 & (be)_3 & (cf)_3 \end{vmatrix} = (ade)\,(bcf) - (adb)\,(ecf)\ ;$$

$$L = \begin{vmatrix} (ab,\ de)_1 & (bc,\ ef)_1 & (ca,\, fd)_1 \\ (ab,\ de)_2 & (bc,\ ef)_2 & (ca,\, fd)_2 \\ (ab,\ de)_3 & (bc,\ ef)_3 & (ca,\, fd)_3 \end{vmatrix}$$

$$= \begin{vmatrix} (ab)_1 & (de)_1 & (ca)_1 \\ (ab)_2 & (de)_2 & (ca)_2 \\ (ab)_3 & (de)_3 & (ca)_3 \end{vmatrix} \cdot \begin{vmatrix} (bc)_1 & (ef)_1 & (fd)_1 \\ (bc)_2 & (ef)_2 & (fd)_2 \\ (bc)_3 & (ef)_3 & (fd)_3 \end{vmatrix}$$

$$- \begin{vmatrix} (ab)_1 & (de)_1 & (fd)_1 \\ (ab)_2 & (de)_2 & (fd)_2 \\ (ab)_3 & (de)_3 & (fd)_3 \end{vmatrix} \cdot \begin{vmatrix} (bc)_1 & (ef)_1 & (ca)_1 \\ (bc)_2 & (ef)_2 & (ca)_2 \\ (bc)_3 & (ef)_3 & (ca)_3 \end{vmatrix}.$$

Reducing this still further, we find:

$$\begin{aligned} L &= [(abc)(dea) - \underline{(aba)}(dec)][(bcf)(efd) - (bcd)\underline{(eff)}] \\ &\quad - [(abf)\underline{(ded)} - (abd)(def)][\underline{(bcc)}(efa) - (bca)(efc)] \\ &= (abc)(def)[(ade)(bcf) - (adb)(ecf)] \\ &= P \cdot (abc)(def). \end{aligned}$$

For the underscored determinants vanish identically. Hence P is a factor of L, and the latter becomes zero with the former,— the fact which was to be proved.

Noticing the other factors of L, we observe that L vanishes also without P when either the three points $A\ B\ C$ or the three points $D\ E\ F$ are collinear; as is evident geometrically, for when the three lines of either triangle coincide, their three intersections with the sides of the other triangle must lie in this triply-counting line.

3. The inscribed hexagon,—Pascal's Theorem. Pascal's theorem may be stated in this form:

If the conic through five vertices of a hexagon contains also the sixth vertex, then the three intersections of pairs of opposite sides of the hexagon will lie in a right line, and conversely.

For proof we shall write down the condition that the three points lie on a line, and show that it is identically the condition that the sixth vertex may lie on the conic determined by the other five.

Designate the vertices by A, B, C, D, E, X, with coordinates respectively (a_1, a_2, a_3), (b_1, b_2, b_3) $\cdots\cdots$ (x_1, x_2, x_3). Pairs of opposite sides are the lines XA, CD; AB, DE; BC, EX. For their three points of intersection to lie upon a line, the necessary and sufficient condition is the vanishing of a determinant, namely:

$$\begin{vmatrix} (xa,\ cd)_1 & (ab,\ de)_1 & (bc,\ ex)_1 \\ (xa,\ cd)_2 & (ab,\ de)_2 & (bc,\ ex)_2 \\ (xa,\ cd)_3 & (ab,\ de)_3 & (bc,\ ex)_3 \end{vmatrix} \equiv K = 0.$$

The equation $K=0$, is of second degree in the coordinates of X; hence the condition is satisfied for all points X of a certain conic. *This conic is the one which contains the five other points, for K vanishes when X coincides with either A, B, C, D, or E.*

(1) When X coincides with A or E, line-coordinates $(xa)_1$, $(xa)_2$, $(xa)_3$ in the first column of K or $(ex)_1$, $(ex)_2$, $(ex)_3$ in the last column will all vanish, whence $K=0$.

(2) When X coincides with B or D, say with B; expand the determinant K, distributing the constituents of the second column. After replacing x_1, x_2, x_3 by b_1, b_2, b_3 we find:

$$K=\begin{vmatrix} (ba)_1 & (cd)_1 & (de)_1 \\ (ba)_2 & (cd)_2 & (de)_2 \\ (ba)_3 & (cd)_3 & (de)_3 \end{vmatrix} \cdot \begin{vmatrix} (ab)_1 & (bc)_1 & (eb)_1 \\ (ab)_2 & (bc)_2 & (eb)_2 \\ (ab)_3 & (bc)_3 & (eb)_3 \end{vmatrix}$$

$$-\begin{vmatrix} (ba)_1 & (cd)_1 & (ab)_1 \\ (ba)_2 & (cd)_2 & (ab)_2 \\ (ba)_3 & (cd)_3 & (ab)_3 \end{vmatrix} \cdot \begin{vmatrix} (de)_1 & (bc)_1 & (eb)_1 \\ (de)_2 & (bc)_2 & (eb)_2 \\ (de)_3 & (bc)_3 & (eb)_3 \end{vmatrix}.$$

The second of these four determinants vanishes because the three lines concerned meet in one point, B; the third vanishes because two columns differ only in sign. Hence again $K=0$.

(3) When X coincides with C, the first and last columns in K become proportional; for in the first, $(ca, cd)_1 = -(a\,c\,d)c_1$, etc.; and in the last, $(bc, ec)_1 = -(b\,c\,e)c_1$, etc.

Hence the locus of X contains all five other points, and is the conic determined by them, as asserted by the theorem to be proved.

For a class exercise, it is worth while to obtain further Brianchon's theorem by arguing from duality or by exactly parallel reasoning with lines in place of points. The three theorems then gain in interest by being combined in the following:

If a hexagon be inscribed in a conic, its alternate sides intersect in six points which are in corresponding order the vertices of a hexagon circumscribed to a second conic.

[*Postscript:* For a proof of Desargues' theorem closely related to the one given above, see a paper by Hunyady, *Crelle* Vol. 89, p. 79.]

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